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UNIFIED ANALYSIS OF SLUG TESTS INCLUDING NONLINEARITIES, INERTIAL EFFECTS, AND TURBULENCE

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Abstract

Slug tests are frequently used to characterize the transmissivity of an aquifer. In highly permeable aquifers, however, problems arise when conventional analytical techniques are applied. In an aquifer consisting of coarse sand and gravel overlain by silt and clay, we have consistently seen deviations from the expected response of linear theoretical models. Typically, we see a systematic lack of fit to traditional models and a dramatic dependence of the slug test on the magnitude of the initial displacement. In some wells we have also observed oscillatory behavior. Although there are some theories describing oscillatory behavior in slug tests, until now it has been difficult to analyze tests which are in the so-called "critically damped" region. We have developed a unified model for slug tests that includes the effect of nonlinear terms, inertia, turbulence (spatial velocity distributions), viscosity and differing casing and screen radii. The equation for the borehole can be obtained by either considering the mechanical energy balance equation or by considering the Navier-Stokes equation. This borehole equation is coupled to the aquifer equation by the boundary conditions at the well screen. Generally, the effects of viscosity and changing casing-screen radii are negligible. However, the effects of nonlinearities, inertia, and spatial velocity distributions can be quite important. The nonlinear terms make slug test results dependent on the initial head, inertial effects are important when oscillatory behavior is observed, and spatial velocity distributions cause the effective water column length to be greater than expected. This general model can be reduced to a Hvorslev type model by assuming no storage in the aquifer. We have obtained an iterative numerical solution to this model and have applied it to field data from our research site. The results are quite good both for oscillatory and non-oscillatory situations and give consistent estimates of the physical parameters for various initial displacements.

Introduction

Slug tests are frequently used to characterize the transmissivity of an aquifer. In highly permeable aquifers, however, problems arise when conventional analytical techniques are applied. At one of our field sites in an aquifer consisting of coarse sand and gravel overlain by silt and clay (GEMS - Geohydrologic Experimental and Monitoring Site), we have consistently seen deviations from the expected response of linear theoretical models. Typically, we see a systematic lack of fit to traditional models and a dramatic dependence of the slug test on the magnitude of the initial displacement (Figures 1 and 2). The transient pressure spike seen at very early time is caused by a water hammer effect due to initiation of the slug test and will be ignored in this paper.

Figure 1 shows some typical slug test data from a GEMS well that does not oscillate, but for which the conventional theories do not offer an adequate explanation. The main problems shown in the data of Figure 1 are: 1. the response is dependent on the initial head and 2. the Hvorslev (1951) and Cooper, Bredehoeft, and Papadopulos (CBP,1967) models show a systematic lack of fit. In all linear theories the normalized responses for various initial slug heights should collapse onto one curve. Clearly, this is not the case in Figure 1.

In some wells we have also observed oscillatory behavior (Figure 2). Some of the earliest attempts to analyze oscillatory data were by Krauss (1974) and van der Kamp (1976), in which they invoked a number of assumptions to make the theory linear. Kipp (1985) has also dealt with the linear theory of oscillatory slug test responses by using Laplace transforms and numerical inversions to calculate type curves. Kabala et al. (1985) are among the first to consider the use of a nonlinear equation to describe the oscillatory slug test behavior.

However, after considerable numerical study, they state that "the linear model is sufficiently accurate in all practical cases." The data in Figure 2 show that their conclusion is not valid for this well.

We have developed a unified model for slug tests that includes the effects of nonlinear terms, inertia, turbulence (spatial velocity distributions), viscosity and differing casing and screen radii. We have developed a numerical solution under Hvorslev type assumptions that should be valid over the whole range from "overdamped" to "underdamped" conditions.

Navier-Stokes Equation for the Borehole

The motion of the water in the borehole can be described by the Navier-Stokes equations (Eskinazi, 1967). If we consider the borehole as a stream tube with average flow in the z direction the z component equation is

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial z} = -g - \frac{\nabla P}{\rho} + \frac{\mu}{\rho}\nabla^2 V \qquad (1)$$

V is the average velocity of the water in the borehole in the z direction, g is the acceleration of gravity, P is the pressure, ρ is the density and μ is the viscosity. This equation is basically a force balance equation per unit fluid mass and can be integrated in the z direction over the length of the borehole shown in Figure 3 to obtain an energy or work balance equation. We assume that the length of the screen (b) is negligible in comparison with the water column length,

$$b << z_o + h(t) \qquad (2)$$

We will also assume that the water is incompressible (ρ is constant) and that the viscosity is constant. P_s and P_a are taken to be the pressures at the screen and the top of the water column, respectively. r_c and r_s are the casing and screen radii, respectively. If we assume a parabolic distribution of velocities across the borehole radius as shown in Figure 4, we can write

$$V = V_o\left[1 - \frac{r^2}{r_c^2}\right]. \tag{3}$$

With all of these definitions and assumptions equation (1) becomes

$$(h+z_o+b)\frac{d^2h}{d^2t} + \frac{1}{2}\left[1 - \left(\frac{r_c^2}{2r_s b}\right)^2\right]\left(\frac{dh}{dt}\right)^2 = -g(h+z_o+b) + \frac{P_s - P_a}{\rho} - \frac{8\mu}{r_c^2\rho}(h+z_o+b)\frac{dh}{dt} \tag{4}$$

which is a nonlinear ordinary differential equation for the time-dependent height of the water column in the borehole.

Hvorslev Style Approximation

In the spirit of the Hvorslev (1951) and Bouwer and Rice (1976) methods, we can assume that the storage in the aquifer is negligible. We shall use the usual definition of the Hvorslev time lag

$$t_o = \frac{\pi r_c^2}{FK} \tag{5}$$

(F is the Hvorslev form factor) and define two more quantities

$$A = \frac{\left[1 - \left(\frac{r_c^2}{2r_s b}\right)^2\right]}{2g\pi r_c^2} \tag{6}$$

and

$$M = \frac{8\mu}{g t_o r_c^2 \rho} \tag{7}$$

With these definitions and dividing by gt_o, equation (4) can be rewritten into an equation embodying the Hvorslev approximation

$$\frac{(h + z_o + b)}{g t_o}\frac{d^2 h}{dt^2} + FKA\left(\frac{dh}{dt}\right)^2 + \left[M(h + z_o + b) + 1\right]\left(\frac{dh}{dt}\right) + \frac{h}{t_o} = 0 \tag{8}$$

Equation (8) is the nonlinear equivalent to the usual linear Hvorslev equation. This equation only has one unknown parameter, K. The rest of the physical parameters in equation (8) can be measured directly in the field or laboratory. Therefore, a least squares fit of the numerical solution of equation (8) to field data for h(t) should yield a value for K, the aquifer conductivity. When the acceleration term is negligible and M = 0 this is the same model presented by McElwee et al.(1992) for the non-oscillating case.

Initial Analysis

Since the fully nonlinear equation (8) can not be solved analytically, we must resort to numerical techniques. We have had good results applying a point iterative method. This numerical solution has been incorporated into an automated well test analysis package called SUPRPUMP (Bohling and McElwee, 1992). As mentioned earlier there is really only one parameter available for fitting in equation (8): K, the hydraulic conductivity of the aquifer. We discovered a number of things when we tried to fit the field data. First of all, it was impossible to fit the overall shape of the oscillatory field data with only one available parameter. The values of A and M in equations (6) and (7) were quite small and did not seem to fit the field data. The value of A calculated from equation (6) for our field data was about .7 sec^2/ft^3. The kinematic viscosity (μ/ρ) is about 10^{-5} ft^2/sec. Therefore, neither of these parameters played an important role in the analysis of our data. We decided to treat A as an adjustable parameter to be determined by fitting the data. McElwee et al. (1992) had pretty good success using this kind of model when no oscillating water column was observed. Unfortunately, when applied to oscillatory data the model with two parameters (A and K) still did not give a good overall fit to the shape of the curve. Most troubling of all, a constant set of values for A and K did not seem to predict the head dependence of the slug test properly. In the process of trying to fit the data, we observed that increasing the length of the water column in the borehole by adding a constant to the term ($h + z_o + b$) in equation (8) led to a much better model reproduction of the general shape of the field data. So, it seemed that something involving acceleratory work was missing in the physical model.

Revision of the Model

An alternate method of deriving the equation of motion of the water column in a slug test can be obtained by considering an energy balance equation (Hansen, 1967). Consider the water column inside the borehole (Figure 3) to be a control volume. The change of energy within the control volume over time is determined by the work done at the free surface and the amount of energy that flows out the screen. Detailed consideration of the average kinetic energy per unit volume of the borehole shows that our earlier equation needs to be modified. In actual fact there will be other velocity components inside the borehole other than the average vertical velocity describing the drop of the water column. These velocity components may be random in nature (turbulence) or axially circular (curl of velocity not zero) but when averaged over the borehole they do not contribute to the net flow of water out the screen. However, these velocity components may carry significant energy and must be considered when averaging the kinetic energy over the control volume, which is the entire borehole. With reasonable assumptions, it can be shown that the average square velocity is larger than the square of the average velocity by a factor greater than one.

$$\overline{V^2} = \left(\frac{dh}{dt}\right)^2 \left[\frac{4}{3} + \alpha^2\right] \tag{9}$$

This implies that the kinetic energy of the water column can be significantly larger than one might suspect based on the average vertical velocity (dh/dt).

Modifying equation (8) as suggested by equation (9) gives the final form for the mathematical model.

$$\frac{(h+z_o+b)}{gt_o}\left[\frac{4}{3}+\alpha^2\right]\frac{d^2h}{dt^2}+FKA\left(\frac{dh}{dt}\right)^2 + \left[M(h+z_o+b)+1\right]\left(\frac{dh}{dt}\right)+\frac{h}{t_o}=0 \qquad (10)$$

By considering an energy based equation we arrive at the same basic equation as is obtained starting from the Navier-Stokes equation; however, the kinetic energy contribution of velocity components other than those in the vertical direction may be considerable and a new model parameter (α) has been added.

Numerical Solution

A numerical method using point iterative techniques can be used to solve equation (10). The numerical expression for the head at the latest time level (n+1) is given by

$$h^{n+1(m+1)}=\frac{coef(n-1,m)h^{n-1}+coef(n)h^n}{coef(n+1,m)} \qquad (11)$$

where m is an iteration index. The coefficients are defined as

$$coef(n+1,m)=1+\left(M+\frac{2\left(\frac{4}{3}+\alpha^2\right)}{gt_o\Delta t}\right)(h^n+z_o+b) + FKA\left(\frac{h^{n+1(m)}-h^{n-1}}{2\Delta t}\right) \quad , \qquad (12)$$

$$coef(n-1,m) = 1 + \left(M - \frac{2\left(\frac{4}{3} + \alpha^2\right)}{g t_o \Delta t} \right)(h^n + z_o + b) + FKA\left(\frac{h^{n+1(m)} - h^{n-1}}{2\Delta t} \right) , \quad (13)$$

and

$$coef(n) = \frac{4\left(\frac{4}{3} + \alpha^2\right)(h^n + z_o + b)}{g t_o \Delta t} - \frac{2\Delta t}{t_o} . \quad (14)$$

Data Analysis

The model represented by equations (11)-(14) has three parameters (α, A, K) which may be adjusted to fit the field data. We have had good results fitting this model to the GEMS data. Figures 5 and 6 show the fitted theoretical values as stars on the field data plots. The theory describes the head dependence and general shape of the field data very well. Both the non-oscillatory (Figure 5) and oscillatory (Figure 6) data are predicted very well with the fitted values. Field data for a variety of initial slug heights are reproduced well for a single set of parameters (α, A, K). Earlier models (McElwee et al., 1992) fit the non-oscillatory data pretty well but the parameters had some dependence on the initial slug height. In general, the effect of the viscosity term in the model appears to be insignificant. The factor ($4/3 + \alpha^2$) in the model implies that velocity components other than in the z direction carry about

15% of the kinetic energy since $\alpha = .5$ is the best fit value. The 4/3 arises from the assumption that there is a parabolic distribution of velocities along the radius. Any other radial distribution will give a slightly different result; however, the important point is that the column is usually carrying more kinetic energy than would be predicted by simply using the average vertical velocity (dh/dt). These two contributions together increase the kinetic energy about 60% over the uniform velocity case. The other parameter, A, was fitted with a magnitude of 55-70 for this field data. This is much larger than would be calculated from equation (6) for A. Clearly, some physical mechanism has been left out of the model, apparently with the same mathematical form as the term involving A in equation (10), but with a much larger magnitude. Further research is needed to shed light on the nature of this mechanism.

Figures 7 and 8 are simply single plots of the field data and theory for one particular value of the initial slug height. These plots allow one to better assess the quality of the fit of theory to experiment. In general the fit is very good. Figure 9 is a Hvorslev type plot of the field data and theory for the non-oscillatory well. Notice that the data is becoming very noisy after about 12 seconds, so little quantitative information is available beyond tht time. Also notice that the theoretical curve is approaching a straight line whose slope is proportional to K at large time (McElwee et al., 1992). However, there is little hope that data could ever be collected in this region since the response is too small. Only in the overdamped case will this straight line portion move into the range where it is measurable. In the case of wells in the critically damped region, we will always see this characteristic downward curvature on a Hvorslev plot.

Conclusions

Generally, the effects of viscosity and changing casing-screen radii are negligible on slug test responses. However, the effects of nonlinearities, inertia, and velocity distributions can be quite important. The nonlinear terms make slug test results dependent on the initial head, inertial effects are important when oscillatory behavior is observed, and non uniform velocity distributions cause the effective water column length to be greater than expected. We have developed a general model incorporating all these features. This general model can be reduced to a Hvorslev type model by assuming no storage in the aquifer. We have obtained an iterative numerical solution to this model and have applied it to field data from our research site. The results are quite good both for oscillatory and non-oscillatory situations and give consistent estimates of the physical parameters for various initial displacements. The theory predicts the general shape and head dependence observed in the field data. Further research in needed to identify the source of the strong nonlinearity represented by one parameter.

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Figure 1.
Slug Test Response at GEMS Well 02

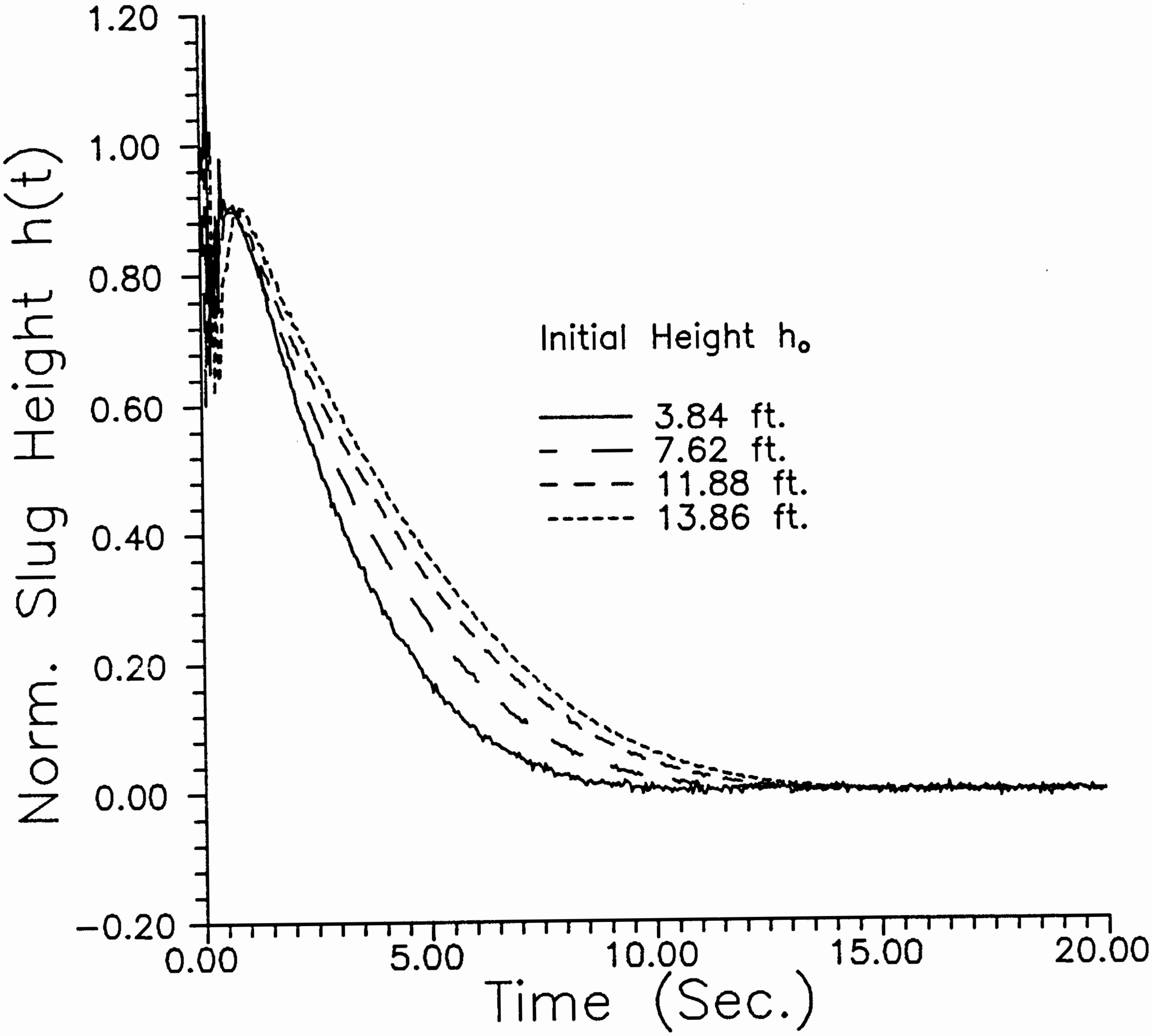


Figure 2.

Slug Test Response at GEMS Well 07

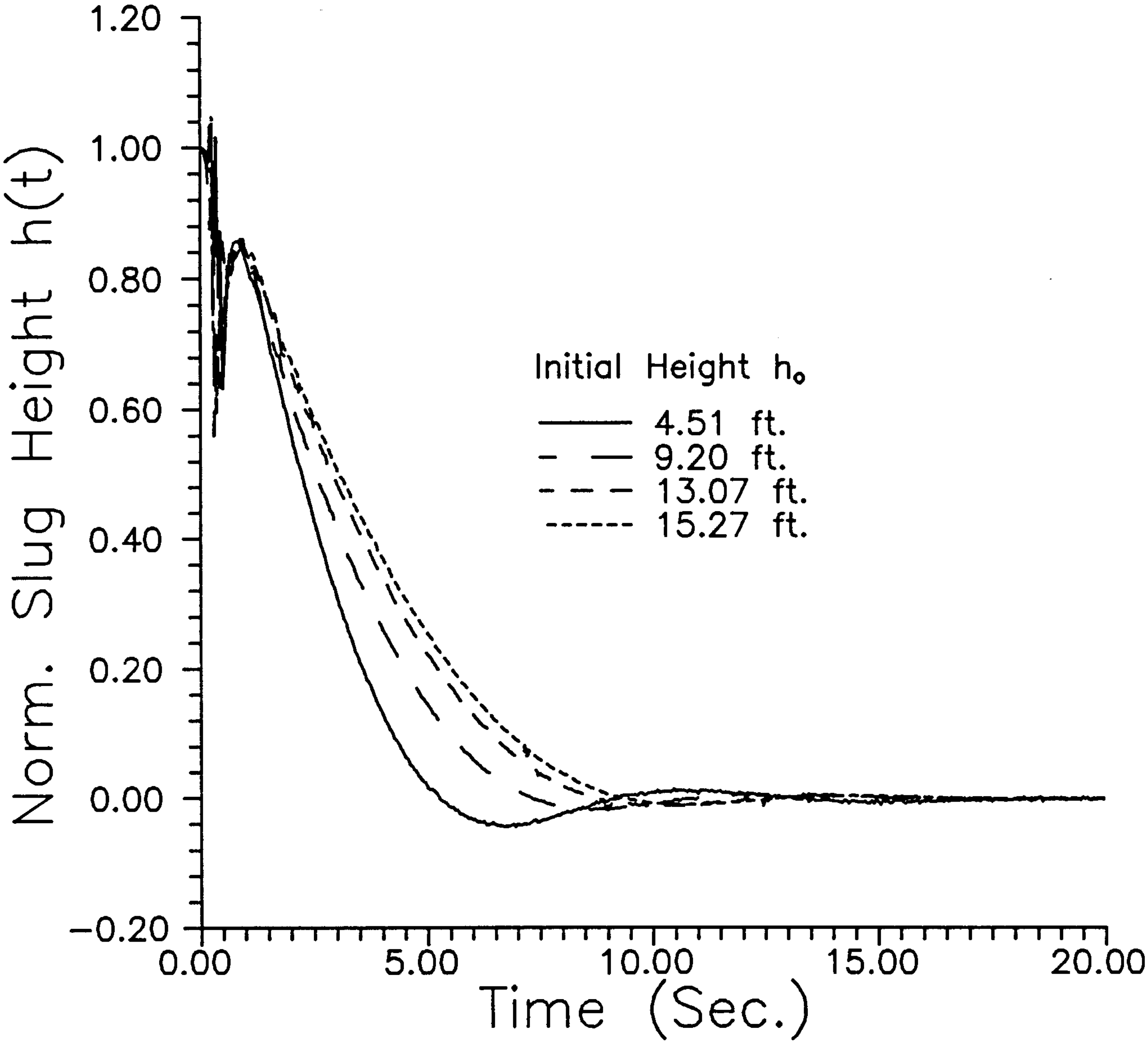
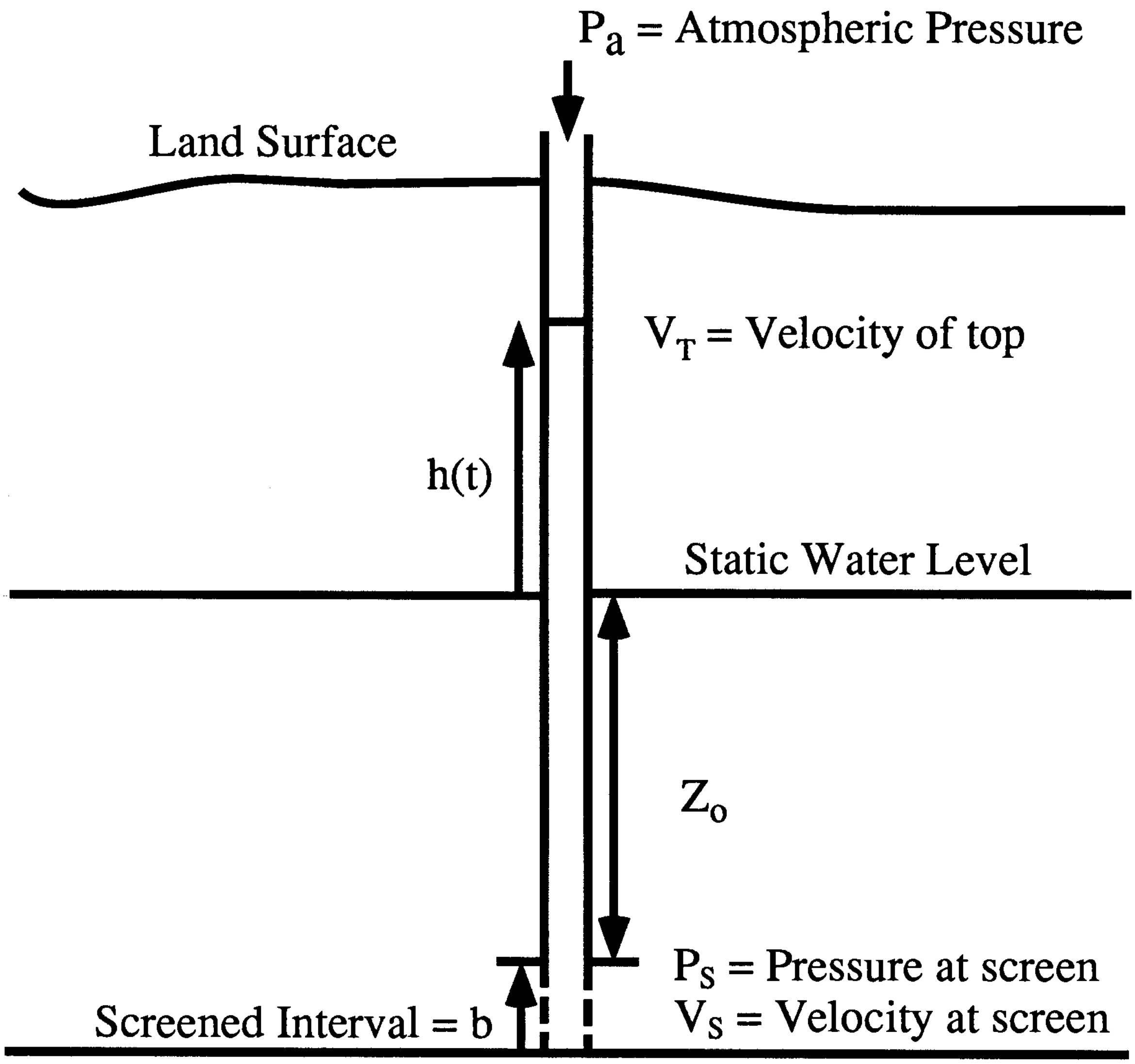


Figure 3. Schematic of the Slug Test Wellbore



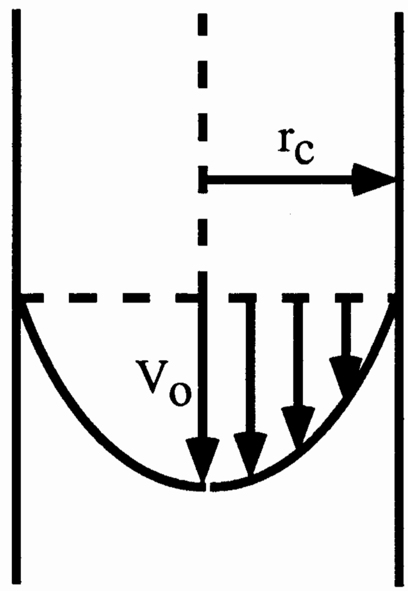


Figure 4. Assumed Radial Velocity Distribution

Figure 5.

Slug Test Response at GEMS Well 02

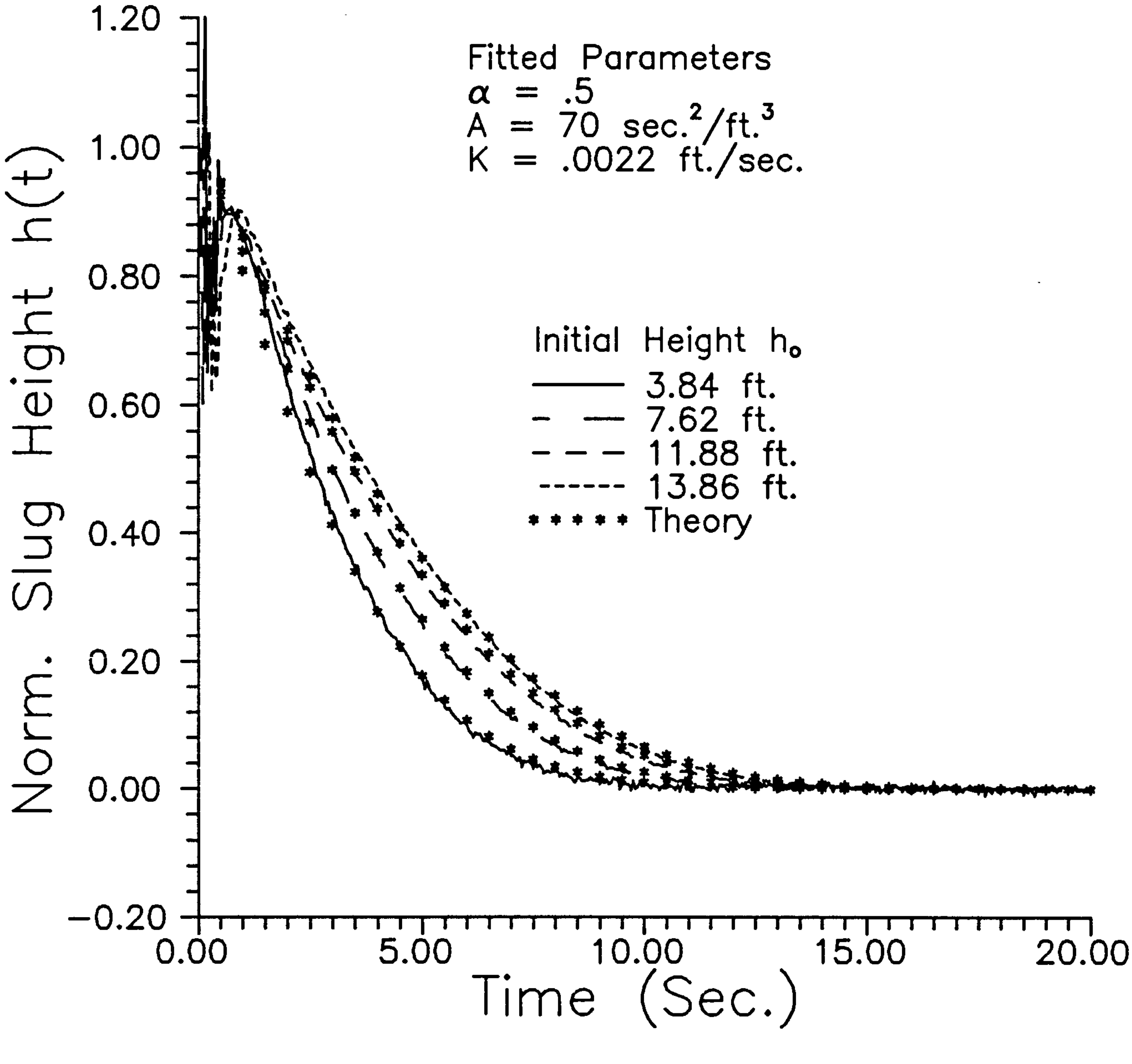


Figure 6.
Slug Test Response at GEMS Well 07

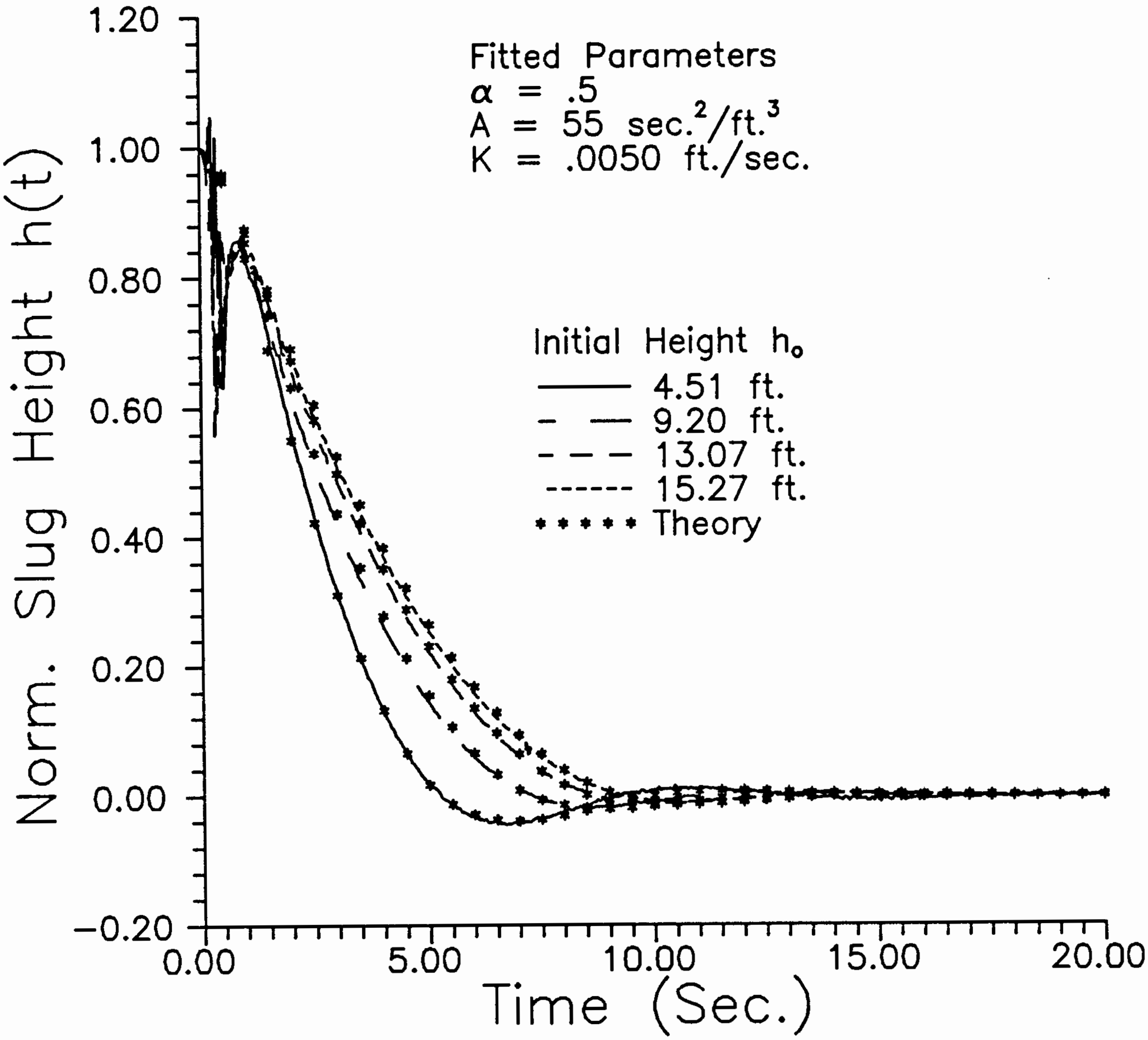


Figure 7.
Slug Test Response at GEMS Well 02

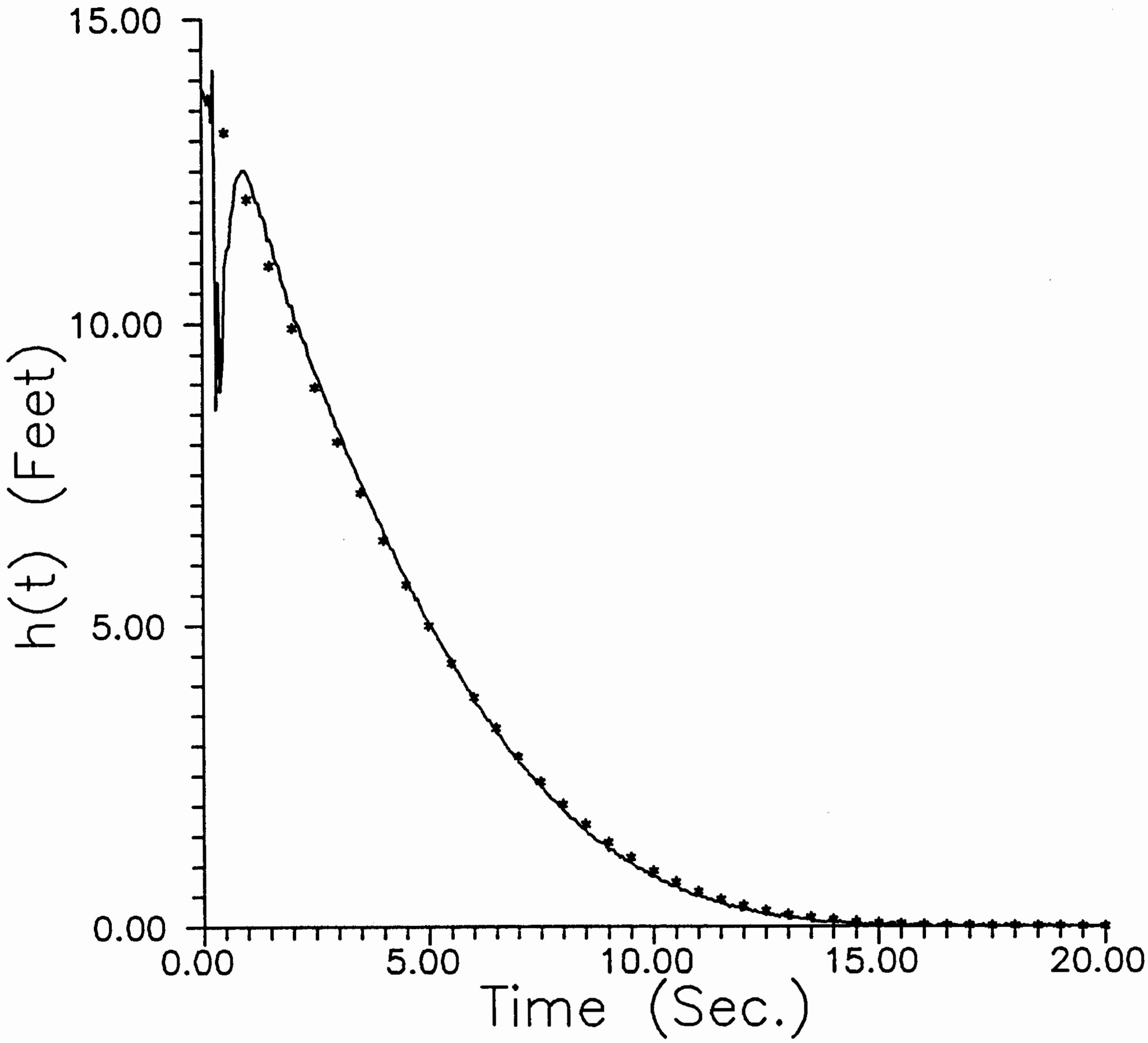


Figure 8.
Slug Test Response at GEMS Well 07

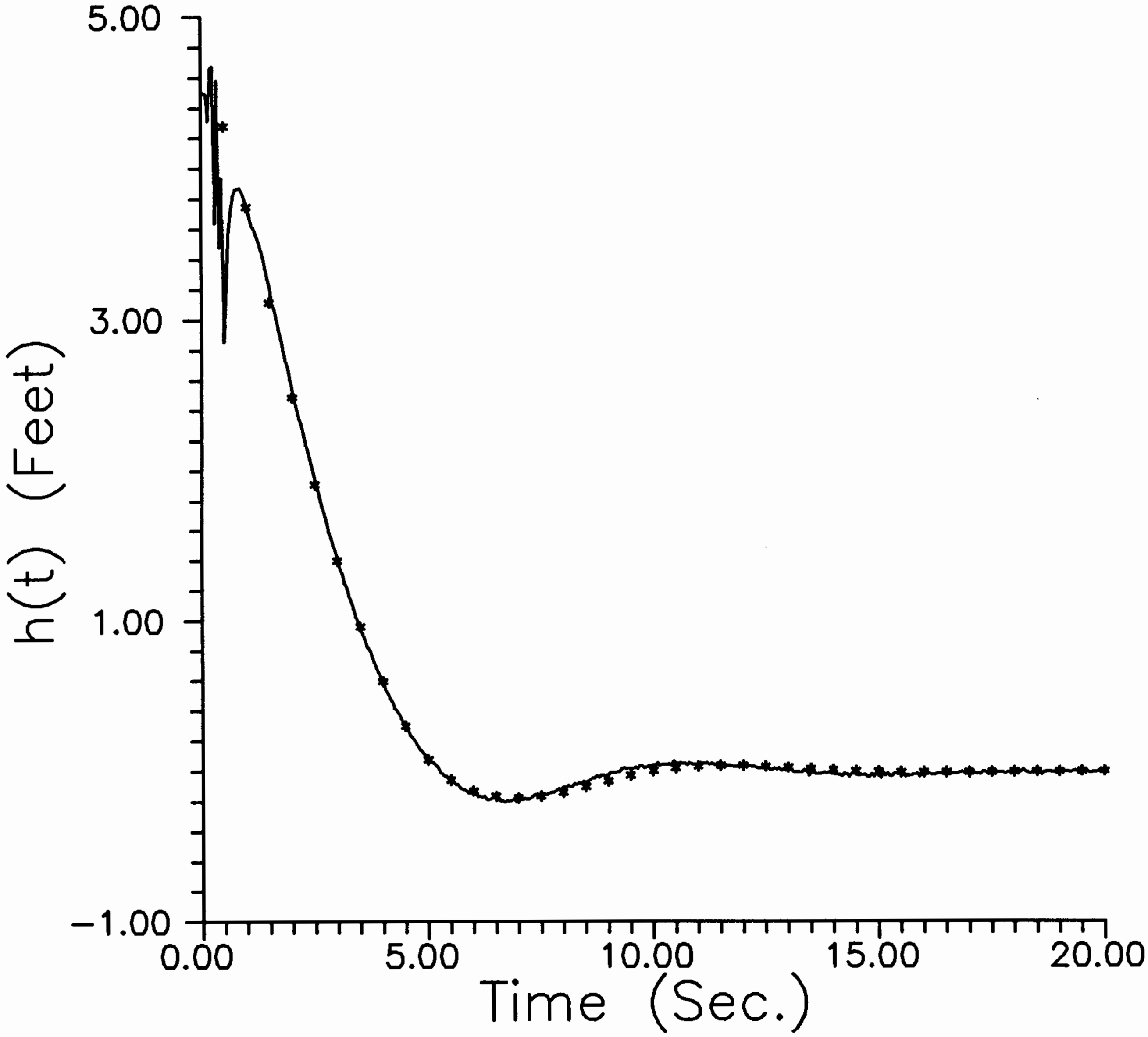


Figure 9.
Slug Test Response at GEMS Well 02