Mathematical Derivations of Semianalytical Solutions for Pumping-Induced Drawdown and Stream Depletion in a Leaky Aquifer System

Xiaoyong Zhan and James J. Butler, Jr.

Kansas Geological Survey
1930 Constant Avenue
University of Kansas
Lawrence, KS. 66047

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I. Introduction

In this report, derivations of the transform-space solutions to the mathematical model describing the drawdown and stream depletion produced by a pumping well in a leaky aquifer system are presented. The back transforms of these expressions are used by *Butler et al.* [in review] to develop new insights into stream-aquifer interactions in a leaky aquifer system.

In Section II of this report, governing equations and auxiliary conditions for a leaky aquifer system hydraulically connected to a stream are presented. The corresponding equations in Laplace-Fourier space are derived in Sections III and IV using standard integral transform methods. The general transform-space solution is presented in Section V. In Sections VI and VII, simplified solutions for an unbounded domain and an unbounded homogeneous domain, respectively, are presented. The derivation of a solution for the stream-depletion rate is presented in Section VIII. The methods used to numerically invert the transform-space solutions are described briefly in Section IX.

II. Governing Equations

The problem of interest here is that of the drawdown and stream depletion produced by pumping from a fully penetrating well in the leaky aquifer system of Figure 1. Following the approach of *Butler et al.* [2001], vertical flow within the upper aquifer is neglected (Dupuit assumptions). The stream and upper aquifer are separated by a zone of relatively low hydraulic conductivity, which is represented mathematically as an incompressible layer (*Hantush* [1965]). Portions of the upper aquifer underneath the stream are confined, but can be confined or unconfined elsewhere. Flow in the aquitard is incorporated using the model of *Hantush* [1960], which includes aquitard storage but neglects lateral flow. Similar to *Hantush and Jacob* [1955], the underlying (lower) aquifer is assumed to be a unit of relatively high permeability so that heads within that aquifer are unaffected by pumping in the upper aquifer. Hydraulic properties are assumed to be a function of x, but can be linearized into a series of zones of uniform properties that are arranged parallel to the stream. Any number of zones can be considered in this derivation but three are used in *Butler et al.* [in review].

Following the approach of *Butler et al.* [2001], governing equations and auxiliary conditions can be defined for the leaky aquifer system of Fig. 1.

Aquifer Flow

$$\begin{aligned} & \frac{\partial^2 s(x,y,t)}{\partial x^2} + \frac{\partial^2 s(x,y,t)}{\partial y^2} - \frac{k_{sb}}{b_{sb}T(x)} s(x,y,t)\left(\mathcal{H}(x-x_{sl}) - \mathcal{H}(x-x_{sr})\right) \\ - & \frac{k_c(x)}{T(x)} \left.\frac{\partial s_c(x,y,z,t)}{\partial z}\right|_{z=0} + \frac{Q}{T(x)}\delta(x-x_p)\delta(y) \\ = & \frac{S(x)}{T(x)}\frac{\partial s(x,y,t)}{\partial t} \qquad x_{lb} < x < x_{rb}, \qquad -\infty < y < \infty, \qquad t > 0 \end{aligned} \tag{1}$$

Note that Eq. (1) is a condensed but generalized form of Eq. (1)-(3) in *Butler et al.* [in review].

Initial condition,

$$s(x,y,t=0) = 0 \tag{2}$$

Left boundary condition in x can be either constant head (Dirichlet condition),

$$\left. s(x,y,t)\right|_{x=x_{lb}} = 0 \tag{3}$$

or no flow (Neumann condition),

$$\left.\frac{\partial s(x,y,t)}{\partial x}\right|_{x=x_{lb}} = 0 \tag{4}$$

Similarly, right boundary condition in x can be,

$$s(x,y,t)\Big|_{x=x_{rb}} = 0 \tag{5}$$

or

$$\frac{\partial s(x,y,t)}{\partial x}\Big|_{x=x_{rb}} = 0 \tag{6}$$

Drawdown at infinity in y is bounded,

$$s(x,y=-\infty,t) < \infty \tag{7}$$

$$s(x,y=\infty,t) < \infty \tag{8}$$

A Cauchy boundary condition could also be readily incorporated into this development. However, for large x_{lb} and x_{rb}, the solution is not sensitive to the form of the lateral boundary conditions.

Aquitard

$$\frac{\partial^2 s_c(x,y,z,t)}{\partial z^2} = \frac{S_{Sc}(x)}{k_c(x)}\frac{\partial s_c(x,y,z,t)}{\partial t}$$
$$x_{lb} < x < x_{rb}, \qquad -\infty < y < \infty, \qquad -b_c < z < 0, \qquad t > 0 \tag{9}$$

Eq. (9) is equivalent to Eq. (4) in *Butler et al.* [in review].

Initial condition,

$$s_c(x,y,z,t=0) = 0 \tag{10}$$

Continuity condition at interface of aquitard and upper aquifer,

$$s_c(x,y,z=0,t) = s(x,y,t) \tag{11}$$

Constant head at base of aquitard,

$$s_c(x,y,z=-b_c,t) = 0 \tag{12}$$

where
x, y = Cartesian coordinates in lateral plane. The origin of the x-axis can be any arbitrary location (e.g., origin defined at right bank of stream in *Butler et al.* [in review]) and the values increase from left to right. The origin of the y axis is at the pumping well and the values increase upward, $[L]$;
z = vertical distance from bottom of upper aquifer, $[L]$;
t = time, $[T]$;

$s(x, y, t)$ = drawdown in the upper aquifer, $[L]$;
$T(x)$ = transmissivity of the upper aquifer, $[L]$;
$S(x)$ = specific yield or storativity of the upper aquifer, [1];
k_{sb} = hydraulic conductivity of streambed, $[L/T]$;
b_{sb} = streambed thickness, $[L]$;
$\mathcal{H}(x - x_{sl})$ = Heaviside function ($= 0$ for $x - x_{sl} < 0$, $= 1$ for $x - x_{sl} > 0$), respectively;
x_{sl}, x_{sr} = left and right boundary of the stream, respectively $[L]$;
$s_c(x, y, z, t)$ = drawdown in the aquitard, $[L]$;
$S_{Sc}(x)$ = specific storage of the aquitard, $[L^{-1}]$;
$k_c(x)$ = hydraulic conductivity of aquitard, $[L/T]$;
$b_c(x)$ = thickness of aquitard, $[L]$;
x_{lb}, x_{rb} = left and right boundary of the aquifer, respectively, $[L]$;
x_p = x coordinate of pumping well, $[L]$;
Q = pumping rate from well located at $(x_p, 0)$, $[L^3/T]$.

A constant rate of pumping is assumed for this development. A variable rate of pumping or a cyclic pumping strategy could be readily incorporated using standard convolution approaches (*Wallace et al.* [1990])

Notation used in this report is the same as that used in the Mathematica package developed for this project and may differ from that used in *Butler et al.* [in review] because of the notation rules for Mathematica and the more generalized form of this development.

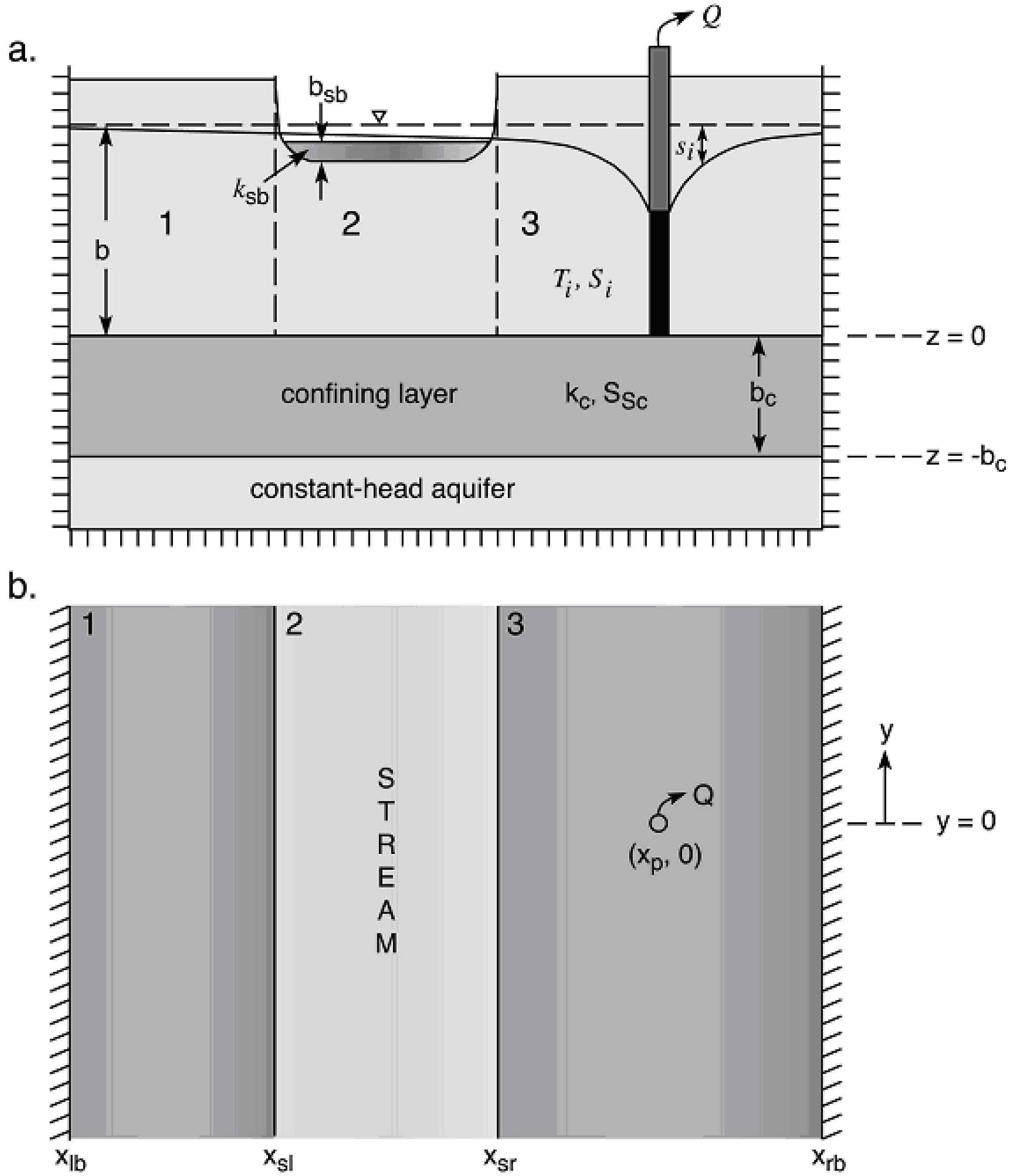

Figure 1: Schematic (a) cross-sectional and (b) areal views of the stream-aquifer system considered in this paper (notation explained in text; stream depletion in this configuration consists of vertical leakage across the low-permeability streambed).

III. Laplace Space Equations

Applying a Laplace transform in t to the equations of the previous section yields

$$\begin{aligned}
&\frac{\partial^2 \overline{s}(x,y)}{\partial x^2} + \frac{\partial^2 \overline{s}(x,y)}{\partial y^2} - \frac{k_{sb}}{b_{sb}T(x)}\overline{s}(x,y)\left(\mathcal{H}(x-x_{sl}) - \mathcal{H}(x-x_{sr})\right) \\
&- \quad \frac{k_c(x)}{T(x)} \left.\frac{\partial \overline{s}_c(x,y,z)}{\partial z}\right|_{z=0} + \frac{Q}{pT(x)}\delta(x-x_p)\delta(y) \\
&= \quad p\frac{S(x)}{T(x)}\overline{s}(x,y) \qquad x_{lb} < x < x_{rb}, \qquad -\infty < y < \infty
\end{aligned} \tag{13}$$

$$\left.\overline{s}(x,y)\right|_{x=x_{lb}} = 0 \qquad or \qquad \left.\frac{\partial \overline{s}(x,y)}{\partial x}\right|_{x=x_{lb}} = 0 \tag{14}$$

$$\left.\overline{s}(x,y)\right|_{x=x_{rb}} = 0 \qquad or \qquad \left.\frac{\partial \overline{s}(x,y)}{\partial x}\right|_{x=x_{rb}} = 0 \tag{15}$$

$$\overline{s}(x, y=-\infty) < \infty \qquad and \qquad \overline{s}(x, y=\infty) < \infty \tag{16}$$

$$\begin{aligned}
&\frac{\partial^2 \overline{s}_c(x,y,z)}{\partial z^2} = p\frac{S_{Sc}(x)}{k_c(x)}\overline{s}_c(x,y,z) \\
&\qquad x_{lb} < x < x_{rb}, \qquad -\infty < y < \infty, \qquad -b_c < z < 0
\end{aligned} \tag{17}$$

$$\overline{s}_c(x,y,z=0) = \overline{s}(x,y) \tag{18}$$

$$\overline{s}_c(x,y,z=-b_c) = 0 \tag{19}$$

where $\overline{s}$ and $\overline{s_c}$ are the Laplace transform of s and s_c, respectively, and p is the Laplace transform parameter. Note the overbar is used to indicate the dependence of s and s_c on p.

After applying the boundary conditions, the general solution for s_c at any z and its first derivative at $z = 0$ are, respectively,

$$\begin{aligned}
\overline{s}_c(x,y,z) \quad &= \cosh\left[\sqrt{\frac{S_{Sc}(x)p}{k_c(x)}}z\right]\overline{s}(x,y) \\
&+ \coth\left[\sqrt{\frac{S_{Sc}(x)p}{k_c(x)}}\right]\sinh\left[\sqrt{\frac{S_{Sc}(x)p}{k_c(x)}}z\right]\overline{s}(x,y)
\end{aligned} \tag{20}$$

and

$$\left.\frac{\partial \overline{s}_c(x,y,z)}{\partial z}\right|_{z=0} = \sqrt{\frac{S_{Sc}(x)p}{k_c(x)}} \coth\left[\sqrt{\frac{S_{Sc}(x)p}{k_c(x)}}\right] \overline{s}(x,y) \tag{21}$$

Substituting the above equation into equation (13) produces,

$$\begin{aligned}
& \frac{\partial^2 \overline{s}(x,y)}{\partial x^2} + \frac{\partial^2 \overline{s}(x,y)}{\partial y^2} - \frac{k_{sb}}{b_{sb}T(x)} \left[\mathcal{H}(x - x_{sl}) - \mathcal{H}(x - x_{sr})\right] \overline{s}(x,y) \\
- & \frac{1}{T(x)} \sqrt{\frac{S_{Sc}(x)p}{k_c(x)}} \coth\left[\sqrt{\frac{S_{Sc}(x)p}{k_c(x)}}\right] \overline{s}(x,y) + \frac{Q}{pT(x)} \delta(x - x_p)\delta(y) \\
= & \; p\frac{S(x)}{T(x)} \overline{s}(x,y) \qquad x_{lb} < x < x_{rb}, \qquad -\infty < y < \infty
\end{aligned} \tag{22}$$

IV. Fourier-Laplace Space Equations

Applying a Fourier transform with respect to y to (22) and (14)-(16) produces:

$$\begin{aligned} &\frac{d^2\tilde{\bar{s}}(x)}{dx^2} - \omega^2\tilde{\bar{s}}(x) - \frac{k_{sb}}{b_{sb}T(x)}[\mathcal{H}(x-x_{sl}) - \mathcal{H}(x-x_{sr})]\,\tilde{\bar{s}}(x) \\ - &\frac{1}{T(x)}\sqrt{\frac{S_{Sc}(x)p}{k_c(x)}}\coth\left[\sqrt{\frac{S_{Sc}(x)p}{k_c(x)}}\right]\tilde{\bar{s}}(x) + \frac{Q}{\sqrt{2\pi}pT(x)}\delta(x-x_p) \\ = &\; p\frac{S(x)}{T(x)}\tilde{\bar{s}}(x) \qquad x_{lb} < x < x_{rb} \end{aligned} \tag{23}$$

$$\tilde{\bar{s}}(x)\Big|_{x=x_{lb}} = 0 \qquad or \qquad \frac{d\tilde{\bar{s}}(x)}{dx}\Big|_{x=x_{lb}} = 0 \tag{24}$$

$$\tilde{\bar{s}}(x)\Big|_{x=x_{rb}} = 0 \qquad or \qquad \frac{d\tilde{\bar{s}}(x)}{dx}\Big|_{x=x_{rb}} = 0 \tag{25}$$

where $\tilde{\bar{s}}$ is the Fourier-Laplace transform of s, and ω is the Fourier transform variable. Note the double overbar is used to indicate the dependence of s on p and ω.

By defining,

$$\begin{aligned} &\beta^2 = \omega^2 + \frac{k_{sb}}{b_{sb}T(x)}[\mathcal{H}(x-x_{sl}) - \mathcal{H}(x-x_{sr})] \\ + &\frac{1}{T(x)}\sqrt{\frac{S_{Sc}(x)p}{k_c(x)}}\coth\left[\sqrt{\frac{S_{Sc}(x)p}{k_c(x)}}\right] + p\frac{S(x)}{T(x)} \end{aligned} \tag{26}$$

The above equation is simplified to,

$$\frac{d^2\tilde{\bar{s}}(x)}{dx^2} - \beta^2\tilde{\bar{s}}(x) + \frac{Q}{\sqrt{2\pi}pT(x)}\delta(x-x_p) = 0 \qquad x_{lb} < x < x_{rb} \tag{27}$$

The above equation is an ordinary differential equation (ODE) that is readily solved in the following section.

V. Transform-Space Solution

Dividing the aquifer into n zones of homogeneous properties in the x direction enables Eq. (27) to be written for each zone i as

$$\frac{d^2\widetilde{\overline{s}}_i(x)}{dx^2} - \beta_i^2\widetilde{\overline{s}}_i(x) + \frac{Q}{\sqrt{2\pi}pT(x)}\delta(x-x_p) = 0 \qquad x_i < x < x_{i+1}, \qquad i = 1,...n \quad (28)$$

Because $\delta(x - x_p)$ vanishes when x is not equal to x_p, the general solution for the above equation is,

$$\widetilde{\overline{s}}_i(x) = c_{2i-1}\sinh(\beta_i x) + c_{2i}\cosh(\beta_i x) \qquad x_i < x < x_{i+1}, \qquad i = 1,...n \quad (29)$$

for bounded zones or

$$\widetilde{\overline{s}}_i(x) = c_{2i-1}e^{\beta_i x} + c_{2i}e^{-\beta_i x} \qquad x_1 \to -\infty, \qquad x_{n+1} \to \infty \quad (30)$$

for unbounded zones. In all cases, the constant coefficients c_i are determined from the boundary conditions.

For external lateral boundary conditions, Eq. (24) and (25) are applied to x_1 at the left boundary and to x_{n+1} at the right boundary, respectively. For inter-boundary conditions at x_i $(i = 2,...n)$, integrating Eq. (28) with respect to x across narrow regions containing x_i $(i = 2,...n)$ reveals that $\widetilde{\overline{s}}_i(x)$ and its first derivative are continuous at all x_i $(i = 2,...n)$ except at x_p $(i = i_p)$ where the first derivative has a discontinuity because of the singular point at the pumping well. The boundary conditions can thus be written as,

$$\left.\begin{array}{l} Dirichlet: \widetilde{\overline{s}}_i(x_i) \\ Neumann: \left.\frac{d\widetilde{\overline{s}}_i(x)}{dx}\right|_{x=x_i} \end{array}\right\} = 0, \qquad i = 1$$

$$\left(\widetilde{\overline{s}}_i(x_{i+1}) - \widetilde{\overline{s}}_{i+1}(x_{i+1})\right) = 0, \qquad i = 1,...,n-1$$

$$\left.\frac{d\left(\widetilde{\overline{s}}_i(x) - \widetilde{\overline{s}}_{i+1}(x)\right)}{dx}\right|_{x=x_{i+1}} = \begin{cases} \frac{Q}{\sqrt{2\pi}pT(x)}, & i = i_p - 1 \\ 0, & all\ other\ i = 1,...,n-1 \end{cases}$$

$$\left.\begin{array}{l} Dirichlet: \widetilde{\overline{s}}_i(x_{i+1}) \\ Neumann: \left.\frac{d\widetilde{\overline{s}}_i(x)}{dx}\right|_{x=x_{i+1}} \end{array}\right\} = 0, \qquad i = n \quad (31)$$

Substituting Eq. (29) into Eq. (31) produces,

$$
\begin{aligned}
&\left.\begin{array}{l} Dirichlet: \sinh(\beta_i x_i)c_{2i-1} + \cosh(\beta_i x_i)c_{2i} \\ Neumann: \cosh(\beta_i x_i)c_{2i-1} + \sinh(\beta_i x_i)c_{2i} \end{array}\right\} = 0, \qquad i = 1 \\
&\quad \sinh(\beta_i x_{i+1})c_{2i-1} + \cosh(\beta_i x_{i+1})c_{2i} \\
&- \quad \sinh(\beta_{i+1} x_{i+1})c_{2i+1} - \cosh(\beta_{i+1} x_{i+1})c_{2i+2} = 0, \qquad i = 1, ..., n-1 \\
&\quad \beta_i \cosh(\beta_i x_{i+1})c_{2i-1} + \beta_i \sinh(\beta_i x_{i+1})c_{2i} - \beta_{i+1}\cosh(\beta_{i+1} x_{i+1})c_{2i+1} \\
&- \quad \beta_{i+1}\sinh(\beta_{i+1} x_{i+1})c_{2i+2} = \begin{cases} \frac{Q}{\sqrt{2\pi}pT(x)}, & i = i_p - 1 \\ 0, & all\ other\ i = 1, ..., n-1 \end{cases} \\
&\left.\begin{array}{l} Dirichlet: \sinh(\beta_i x_{i+1})c_{2i-1} + \cosh(\beta_i x_{i+1})c_{2i} \\ Neumann: \cosh(\beta_i x_{i+1})c_{2i-1} + \sinh(\beta_i x_{i+1})c_{2i} \end{array}\right\} = 0, \qquad i = n
\end{aligned} \tag{32}
$$

By defining

$$
\sinh(\beta_i x_j) = a_{i,j}, \qquad \cosh(\beta_i x_j) = b_{i,j}, \qquad a_{i,i} = a_i, \qquad b_{i,i} = b_i \tag{33}
$$

Eq. (32) can be rewritten in a $2n \times 2n$ matrix format, as shown in the following for which the left boundary is a Dirichlet condition and the right boundary is a Neumann condition,

$$
\begin{pmatrix}
a_1 & b_1 & & & & & & \\
a_{1,2} & b_{1,2} & -a_2 & -b_2 & & & & \\
\beta_1 b_{1,2} & \beta_1 a_{1,2} & -\beta_2 b_2 & -\beta_2 a_2 & & & & \\
 & \ddots & & & & & & \\
 & & a_{i,i+1} & b_{i,i+1} & -a_{i+1} & -b_{i+1} & & \\
 & & \beta_i b_{i,i+1} & \beta_i a_{i,i+1} & -\beta_{i+1} b_{i+1} & -\beta_{i+1} a_{i+1} & & \\
 & & & & & & \ddots & \\
 & & & & a_{n-1,n} & b_{n-1,n} & -a_n & -b_n \\
 & & & & \beta_{n-1} b_{n-1,n} & \beta_{n-1} a_{n-1,n} & -\beta_n b_n & -\beta_n a_n \\
 & & & & & & b_{n,n+1} & a_{n,n+1}
\end{pmatrix}
$$

$$
\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{2i-1} \\ c_{2i} \\ c_{2i+1} \\ c_{2i+2} \\ \vdots \\ c_{2n-1} \\ c_{2n} \end{pmatrix}
=
\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \begin{cases} \frac{Q}{\sqrt{2\pi}pT(x)}, & i = i_p - 1 \\ 0, & all\ other\ i = 1, ..., n-1 \end{cases} \\ 0 \\ \vdots \\ 0 \end{pmatrix}
$$

The expression for $\widetilde{\overline{s}}_i(x)$ can be obtained by solving the above linear equation for c_i and substituting the solution into Eq. (29).

For $n = 4$ zones with $x = \{x_1, x_2, x_3, x_4, x_5\} = \{x_{lb}, x_{sl}, x_{sr}, x_p, x_{rb}\}$, the above matrix equation can be simplified to,

$$\begin{pmatrix} a_1 & b_1 & & & & & & \\ a_{1,2} & b_{1,2} & -a_2 & -b_2 & & & & \\ \beta_1 b_{1,2} & \beta_1 a_{1,2} & -\beta_2 b_2 & -\beta_2 a_2 & & & & \\ & & a_{2,3} & b_{2,3} & -a_3 & -b_3 & & \\ & & \beta_2 b_{2,3} & \beta_2 a_{2,3} & -\beta_3 b_3 & -\beta_3 a_3 & & \\ & & & & a_{3,4} & b_{3,4} & -a_4 & -b_4 \\ & & & & \beta_3 b_{3,4} & \beta_3 a_{3,4} & -\beta_4 b_4 & -\beta_4 a_4 \\ & & & & & & b_{4,5} & a_{4,5} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{Q}{\sqrt{2\pi} p T(x)} \\ 0 \end{pmatrix}$$

Expressions for c_i and $\widetilde{\widetilde{s}}_i$ are very lengthy and not provided here for the sake of brevity. A Mathematica notebook is available for interested readers. However, the expressions used for all examples in *Butler et al.* [in review] are presented in the Section VII.

VI. Transform-Space Solution for Unbounded Domain

For an unbounded domain in the x direction, $x_{lb} = -\infty$ and $x_{rb} = \infty$, an approach similar to that in Section V is adopted to derive the boundary conditions in which Eq. (30) is used for the unbounded (far left and far right) zones.

$$\begin{aligned}
& c_{2i} = 0, \qquad i = 1 \\
& e^{\beta_i x_{i+1}} c_{2i-1} - \sinh(\beta_{i+1} x_{i+1}) c_{2i+1} - \cosh(\beta_{i+1} x_{i+1}) c_{2i+2} = 0, \qquad i = 1 \\
& \beta_i e^{\beta_i x_{i+1}} c_{2i-1} - \beta_{i+1} \cosh(\beta_{i+1} x_{i+1}) c_{2i+1} \\
- \quad & \beta_{i+1} \sinh(\beta_{i+1} x_{i+1}) c_{2i+2} = \begin{cases} \frac{Q}{\sqrt{2\pi} p T(x)}, & i = i_p - 1 = 1 \\ 0, & i = 1 \neq i_p - 1 \end{cases} \\
& \sinh(\beta_i x_{i+1}) c_{2i-1} + \cosh(\beta_i x_{i+1}) c_{2i} \\
- \quad & \sinh(\beta_{i+1} x_{i+1}) c_{2i+1} - \cosh(\beta_{i+1} x_{i+1}) c_{2i+2} = 0, \qquad i = 2, ..., n-2 \\
& \beta_i \cosh(\beta_i x_{i+1}) c_{2i-1} + \beta_i \sinh(\beta_i x_{i+1}) c_{2i} - \beta_{i+1} \cosh(\beta_{i+1} x_{i+1}) c_{2i+1} \\
- \quad & \beta_{i+1} \sinh(\beta_{i+1} x_{i+1}) c_{2i+2} = \begin{cases} \frac{Q}{\sqrt{2\pi} p T(x)}, & i = i_p - 1 \\ 0, & \textit{all other } i = 2, ..., n-2 \end{cases} \\
& \sinh(\beta_i x_{i+1}) c_{2i-1} + \cosh(\beta_i x_{i+1}) c_{2i} - e^{-\beta_{i+1} x_{i+1}} c_{2i+2} = 0, \qquad i = n-1 \\
& \beta_i \cosh(\beta_i x_{i+1}) c_{2i-1} + \beta_i \sinh(\beta_i x_{i+1}) c_{2i} \\
- \quad & \beta_{i+1} e^{-\beta_{i+1} x_{i+1}} c_{2i+2} = \begin{cases} \frac{Q}{\sqrt{2\pi} p T(x)}, & i = i_p - 1 = n-1 \\ 0, & \textit{all other } i = n-1 \neq i_p - 1 \end{cases} \\
& c_{2i-1} = 0, \qquad i = n
\end{aligned} \tag{34}$$

By applying Eq. (33) and redefining

$$e^{\beta_1 x_2} = a_{1,2} = b_{1,2} \qquad e^{-\beta_n x_n} = a_n = b_n \tag{35}$$

Eq. (34) can be rewritten in a $2n \times 2n$ matrix format, as shown in the following for which left boundary is a Dirichlet condition and the right boundary is a Neumann condition,

$$\begin{pmatrix}
0 & 1 & & & & & & \\
a_{1,2} & 0 & -a_2 & -b_2 & & & & \\
\beta_1 b_{1,2} & 0 & -\beta_2 b_2 & -\beta_2 a_2 & & & & \\
 & \ddots & & & & & & \\
 & & a_{i,i+1} & b_{i,i+1} & -a_{i+1} & -b_{i+1} & & \\
 & & \beta_i b_{i,i+1} & \beta_i a_{i,i+1} & -\beta_{i+1} b_{i+1} & -\beta_{i+1} a_{i+1} & & \\
 & & & & & & \ddots & \\
 & & & & a_{n-1,n} & b_{n-1,n} & 0 & -b_n \\
 & & & & \beta_{n-1} b_{n-1,n} & \beta_{n-1} a_{n-1,n} & 0 & -\beta_n a_n \\
 & & & & & & 1 & 0
\end{pmatrix}
\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{2i-1} \\ c_{2i} \\ c_{2i+1} \\ c_{2i+2} \\ \vdots \\ c_{2n-1} \\ c_{2n} \end{pmatrix}$$

$$
= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \begin{cases} \frac{Q}{\sqrt{2\pi}pT(x)}, & i = i_p - 1 \\ 0, & all\ other\ i = 1, ..., n-1 \end{cases} \\ 0 \\ \vdots \\ 0 \end{pmatrix}
$$

The expression for $\widetilde{\overline{s}}_i(x)$ can be obtained by solving the above linear equation for c_i and substituting the solution into Eq. (29) or (30).

For $n = 4$ zones with $x = \{x_1, x_2, x_3, x_4, x_5\} = \{x_{rl} = -\infty, x_{sl}, x_{sr}, x_p, x_{rb} = \infty\}$, the above matrix equation is written as,

$$
\begin{pmatrix}
0 & 1 & & & & & & \\
a_{1,2} & 0 & -a_2 & -b_2 & & & & \\
\beta_1 b_{1,2} & 0 & -\beta_2 b_2 & -\beta_2 a_2 & & & & \\
 & & a_{2,3} & b_{2,3} & -a_3 & -b_3 & & \\
 & & \beta_2 b_{2,3} & \beta_2 a_{2,3} & -\beta_3 b_3 & -\beta_3 a_3 & & \\
 & & & & a_{3,4} & b_{3,4} & 0 & -b_4 \\
 & & & & \beta_3 b_{3,4} & \beta_3 a_{3,4} & 0 & -\beta_4 a_4 \\
 & & & & & & 1 & 0
\end{pmatrix}
\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{Q}{\sqrt{2\pi}pT(x)} \\ 0 \end{pmatrix}
$$

A Mathematica notebook is available for interested readers with expressions for c_i and $\widetilde{\overline{s}}_i$. Expressions for the simple unbounded and homogeneous system used for all examples in *Butler et al.* [in review] are presented in the next section.

VII. Transform-Space Solution for Unbounded Homogeneous Aquifer

For a homogeneous domain and the zone underneath the stream ($x_{sl} < x < x_{sr}$), Eq. (26) can be simplified to

$$\beta_s^2 = \omega^2 + \frac{k_{sb}}{b_{sb}T} + \frac{1}{T}\sqrt{\frac{S_{Sc}p}{k_c}}\coth\left[\sqrt{\frac{S_{Sc}p}{k_c}}\right] + p\frac{S}{T} \tag{36}$$

Outside that zone, the equation can be further simplified to

$$\beta^2 = \omega^2 + \frac{1}{T}\sqrt{\frac{S_{Sc}p}{k_c}}\coth\left[\sqrt{\frac{S_{Sc}p}{k_c}}\right] + p\frac{S}{T} \tag{37}$$

Given the homogeneous domain, the problem can be reduced to solving a group of linear equations for four zones, the boundaries of which are defined by the pumping well and by the two sides of the stream:

$$\tilde{\tilde{s}}(x) = \begin{cases} \sqrt{\frac{2}{\pi}}Q\beta_s e^{-(x_p-x_{sr}+x_{sl}-x)\beta-(x_{sr}-x_{sl})\beta_s}\frac{1}{D}, & -\infty < x < x_{sl} \\ \frac{1}{\sqrt{2\pi}}Qe^{-(x_p-x_{sr})\beta-(x_{sr}+x-2x_{sl})\beta_s} \\ (e^{2(x-x_{sl})\beta_s}(\beta+\beta_s)-(\beta-\beta_s))\frac{1}{D}, & x_{sl} < x < x_{sr} \\ -\frac{1}{\sqrt{2\pi}}Q\frac{1}{2\beta}e^{-(x_p+x)\beta-2x_{sr}\beta_s} \\ ((e^{2x\beta}+e^{2x_{sr}\beta})(e^{2x_{sr}\beta_s}-e^{2x_{sl}\beta_s})\beta^2 \\ 2e^{2x\beta}(e^{2x_{sr}\beta_s}+e^{2x_{sl}\beta_s})\beta\beta_s \\ (e^{2x\beta}-e^{2x_{sr}\beta})(e^{2x_{sr}\beta_s}-e^{2x_{sl}\beta_s})\beta_s^2)\frac{1}{D}, & x_{sr} < x < x_p \\ -\frac{1}{\sqrt{2\pi}}Q\frac{1}{2\beta}e^{-(x_p+x)\beta-2x_{sr}\beta_s} \\ ((e^{2x_p\beta}+e^{2x_{sr}\beta})(e^{2x_{sr}\beta_s}-e^{2x_{sl}\beta_s})\beta^2 \\ 2e^{2x_p\beta}(e^{2x_{sr}\beta_s}+e^{2x_{sl}\beta_s})\beta\beta_s \\ (e^{2x_p\beta}-e^{2x_{sr}\beta})(e^{2x_{sr}\beta_s}-e^{2x_{sl}\beta_s})\beta_s^2)\frac{1}{D}, & x_p < x < \infty \end{cases} \tag{38}$$

in which

$$D = pT(e^{2(x_{sr}-x_{sl})\beta_s}(\beta-\beta_s)^2-(\beta+\beta_s)^2).$$

For the case of an impermeable formation underlying the shallow aquifer of Fig. 1, the leakage (coth) term in Eq. (36) and (37) is negligible, resulting in:

$$\beta_s^2 = \omega^2 + \frac{k_{sb}}{b_{sb}T} + p\frac{S}{T} \tag{39}$$

for the zone underneath the stream ($x_{sl} < x < x_{sr}$) and

$$\beta^2 = \omega^2 + p\frac{S}{T} \tag{40}$$

for all other zones. The solution is thus the same as Eq. (38) except for the definitions of β and β_s.

VIII. Transform-Space Solution for Stream Depletion

The rate of stream depletion Δq is defined as the total volumetric discharge across the incompressible streambed at any given time (see Fig. 1 and caption). For the simplified scenario of Fig. 1 and the previous section, Δq can be expressed as:

$$\Delta q(t) = \frac{k_{sb}}{b_{sb}} \int_{-\infty}^{\infty} \int_{x_{sl}}^{x_{sr}} s_s dx dy \tag{41}$$

in which s_s is the drawdown beneath the stream.

Application of the Laplace transform to Eq. (41) and switching the x and y integrals yields:

$$\Delta \bar{q}(p) = \frac{k_{sb}}{b_{sb}} \int_{x_{sl}}^{x_{sr}} \int_{-\infty}^{\infty} \bar{s}_s dy dx = \frac{k_{sb}}{b_{sb}} \int_{x_{sl}}^{x_{sr}} \tilde{\bar{s}}_s dx \tag{42}$$

in which $\Delta \bar{q}(p)$ is the Laplace transform of Δq and $\tilde{\bar{s}}_s$ is the Fourier-Laplace transform of s_s for $\omega = 0$:

Substitution of Eq. (38) into Eq. (42) and performing the integration results in:

$$\Delta \bar{q}(p) = -\frac{1}{\sqrt{2\pi}} Q e^{-(x_p - x_{sr})\beta} (e^{x_{sr}\beta_s} - e^{x_{sl}\beta s}) \frac{1}{D_s} \tag{43}$$

in which

$$D_s = pT((e^{x_{sr}\beta_s} + e^{x_{sl}\beta_s})\beta\beta_s + (e^{x_{sr}\beta_s} - e^{x_{sl}\beta_s})\beta_s^2).$$

IX. Numerical Inversion of Laplace-Fourier Space Solutions

The solutions in Laplace-Fourier space given in the previous sections are most readily evaluated using a numerical inversion scheme. A Mathematica Add-On package prepared by *Mallet* [2000] is used for the joint Laplace-Fourier numerical inversion. This package provides five inversion methods to invert Laplace transforms and joint Fourier/Hankel-Laplace transforms. The inversion techniques are those of *Durbin* [1974], *Stehfest* [1970], *Weeks* [1966], *Piessens* [1971], and *Crump* [1976]. The *Stehfest* [1970] algorithm, the most commonly used inversion algorithm for well-hydraulics applications, is selected for the inversion of head responses and stream depletion from Laplace space. The Fourier inversion uses the symbolic inverse Fourier transform provided in Mathematica.

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